THE GEOMETRY OF COTTON TENSOR: CONFORMAL SYMMETRY IN DIMENSION THREE

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Preliminaries

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Lorentzian setting: The manifold is a *pp*-wave.

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and the associated Cotton operator: $\tilde{\mathrm{C}}(x, y) = g(\hat{\mathrm{C}}(x), y)$

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In dimension three, Walker manifolds admit local coordinates (t, x, y) where the metric expresses as

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In the special case when the parallel degenerate line field is spanned by a parallel null vector field, the coordinates above can be further specialize so that the metric takes the above form for some function $f(x, y)$. The Walker metric is said to be *strict* in such a case.

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Let $\{e_1, e_2, e_3\}$ be orthonormal $(+\,+\,-)$ such that $\hat{\mathrm{C}} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{pmatrix}$.

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Since the Cotton operator is parallel, the distribution defined by the eigenspace corresponding to λ is parallel and spacelike and the manifold again decomposes locally as a product.

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Take $\{e_1, e_2, e_3\}$ with $g_{11} = g_{23} = 1$ such that $\hat{\mathrm{C}} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}$.

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If $\lambda \neq 0$, the distribution defined by the eigenspace corresponding to λ is parallel and spacelike and the manifold again decomposes locally as a product. The manifold is locally conformally flat.

If $\lambda = 0$ then $\hat{C}$ is 2-step nilpotent. In this case, $\mathrm{Im}(\hat{C}) = \langle e_2 \rangle$ is one-dimensional, null and parallel. The manifold is Walker.

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In this case, $\hat{\mathrm{C}}$ is 3-step nilpotent. We consider $\mathrm{Ker}(\hat{\mathrm{C}})$, which is one-dimensional, null and parallel.

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In this case, $\hat{\mathrm{C}}$ diagonalizes and thus non-trivial examples do not exist.

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(M^3, g) Riemannian essentially conformally symmetric

In this case, $\hat{\mathrm{C}}$ diagonalizes and thus non-trivial examples do not exist.

(M^3, g) Lorentzian essentially conformally symmetric

Main result

Theorem

(M^3, g) is essentially conformally symmetric if and only if it is a strict Lorentzian Walker manifold, locally isometric to $(\mathbb{R}^3, (t, x, y), g_{\mathfrak{a}})$, where

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It must be a Walker manifold. So there exist local coordinates (t, x, y) such that

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$\nabla\,\hat{\mathrm{C}} = 0$ is equivalent to:

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$$\left\{\begin{array}{rcl} f_{tttt} = f_{tttx} = f_{ttxx} = f_{txxx} & = & 0, \\ f_t f_{ttt} - 2f_{ttty} & = & 0, \\ 2f_{ttxy} - f_x f_{ttt} & = & 0, \\ 4f_{txxy} + (2f_{txx} + f_{tty})\, f_t + 2f_{ttyy} - 3f_x f_{ttx} - f_y f_{ttt} - 2ff_{ttty} & = & 0, \\ (f_{tx})^2 + 2f_{xxxx} + f_t f_{txx} + 2f_{txxy} - f_{xx} f_{tt} - 2f_x f_{ttx} & = & 0, \\ f_{tx}\, (f_t)^2 + (2f_{xxx} + 3f_{txy})\, f_t + 2f_{xxxy} + f_{ty} f_{tx} & & \\ +2f_{txyy} - f_{xy} f_{tt} - (2f_{txx} + f_t f_{tt} + 2f_{tty})\, f_x - (f_y + ff_t)\, f_{ttx} & = & 0. \end{array}\right.$$

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Solution:

$$f(t, x, y) = -\frac{\mathcal{D}'(y)}{\mathcal{D}(y)} t + \mathcal{D}(y)x^3 + \mathcal{C}(y)x^2 + \mathcal{B}(y)x + \mathcal{A}(y)$$

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The Ricci operator, $\hat{\rho}$, satisfies $\hat{\rho}^2 = 0$, which implies that the manifold is strict Walker and therefore there exist local coordinates (t, x, y) such that

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For a metric $g = dt\,dy + dx^2 + f(x, y)dy^2$:

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$$\tilde{\mathsf{C}}(\partial_y, \partial_y) = -\tfrac{1}{2} f_{xxx},$$

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By a direct calculation we obtain:

$$f(x,y) = \kappa x^3 + x^2 \mathcal{A}(y) + \mathcal{B}(y)x + \mathcal{C}(y)$$

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Proceeding as in [E. García-Río, P. Gilkey and S. Nikčević, Homogeneity of Lorentzian three-manifolds with recurrent curvature, *Math. Nachr.* **287** (2014), no. 1, 32-47] it is shown that the above metric is locally isometric to

$$g_\mathfrak{a} = dt\,dy + dx^2 + (x^3 + \mathfrak{a}(y)x)dy^2$$

for a suitable function $\mathfrak{a}(y)$.

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[E. Calviño-Louzao, E. García-Río and R. Vázquez-Lorenzo, A note on compact Cotton solitons, *Classical Quantum Gravity* **29** (2012), 205014]

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Any three-dimensional essentially conformally symmetric pseudo-Riemannian manifold is a steady gradient Cotton soliton such that the gradient of the potential function φ, $\nabla\varphi$, is a null vector field.

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$$\mathfrak{a}(y) = \begin{cases} \frac{\alpha}{(4\gamma - \lambda y)^4} - \frac{3}{\lambda}, & \text{if} \quad \lambda \neq 0, \\ \frac{3}{\gamma} y + \alpha, & \text{if} \quad \lambda = 0, \end{cases}$$

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where α is an arbitrary constant. The Ricci soliton vector field is

$$X(t, x, y) = \left(\frac{5\lambda t}{4} + \kappa, \frac{\lambda x}{2}, \gamma - \frac{\lambda y}{4} \right),$$

where κ is an arbitrary real constant.